

Geometric Pattern and its Geometry, Part 14, Two in One – a case study

Mirosław L. Majewski

mirek.majewski@gmail.com

School of Arts & Sciences, New York Institute of Technology,
Abu Dhabi Campus¹, UAE

Abstract

The concept of embedded and two-level patterns will be discussed. We will show how to develop a geometric pattern with one main pattern and another pattern inserted into the empty areas of the first design. For this study, we will use one of the many wall designs at the Hazrat-e Abdolazim Shrine in Shahr-e Rey, Iran.

Introduction

In [8] we discussed the construction of geometric patterns using alternative tessellations in decagonal geometry. For a pattern designer, this method can be more useful than using traditional tessellations. We can apply color or any other decoration directly to the large pattern's convex elements.

In any geometric pattern, we have two types of shapes – those that are factual elements of the pattern and those that are between them. This last group can be considered as a background for the actual pattern. The following drawings show what it looks like in a concrete design.

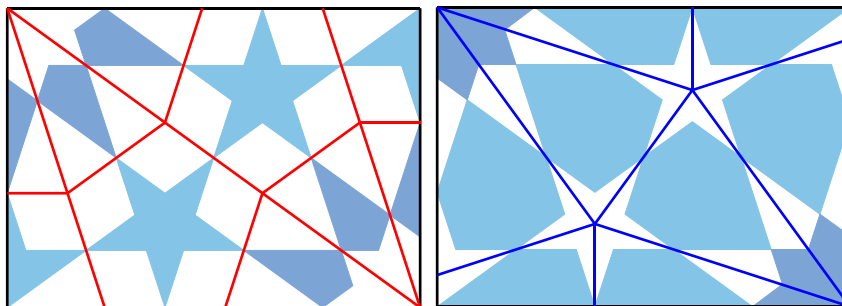


Fig. 1. Two versions of the same pattern

In the left drawing, we used the traditional approach. In the right drawing, we used the alternative tessellation concept. The colored elements in the left image occupy 29.77% of the entire rectangle. In the right image, the colored elements occupy 72.23% of the whole rectangle. This comparison shows that the colored region created in the alternative tessellation is 2.368 times larger than in standard tessellations. Here, we certainly see that these colorful shapes, created through an alternative tessellation, are the visual core of this design. White shapes form the background. Thus, it makes sense to focus on the color elements rather than on the background.

NOTE: Tessellations in the alternative approach were drawn in blue, and the standard tessellations in red. In this paper we continue using this convention.

The alternative tessellations were discussed in [8], where the necessary concepts were outlined. This paper is the promised continuation of the topic of alternative tessellations, with an emphasis on

¹ In 2018, New York Tech announced that it will no longer enroll students at its Abu Dhabi campus. A few years later the Abu Dhabi campus was closed.

optimizing file size in the computer memory. We will concentrate on structures suitable for such a design. We will use computer geometry software *Geometer's Sketchpad*, but one can use *GeoGebra* or follow this paper with *hand-drawn images*.

In Iran, we can find many buildings with wall decorations featuring two layers of patterns. One is the main one (sometimes called primary), and the other one, secondary, filling the primary pattern shapes. Such constructions are often called “two-level patterns”. But this naming is not always correct. In some cases, the term “nested pattern” could be more correct.

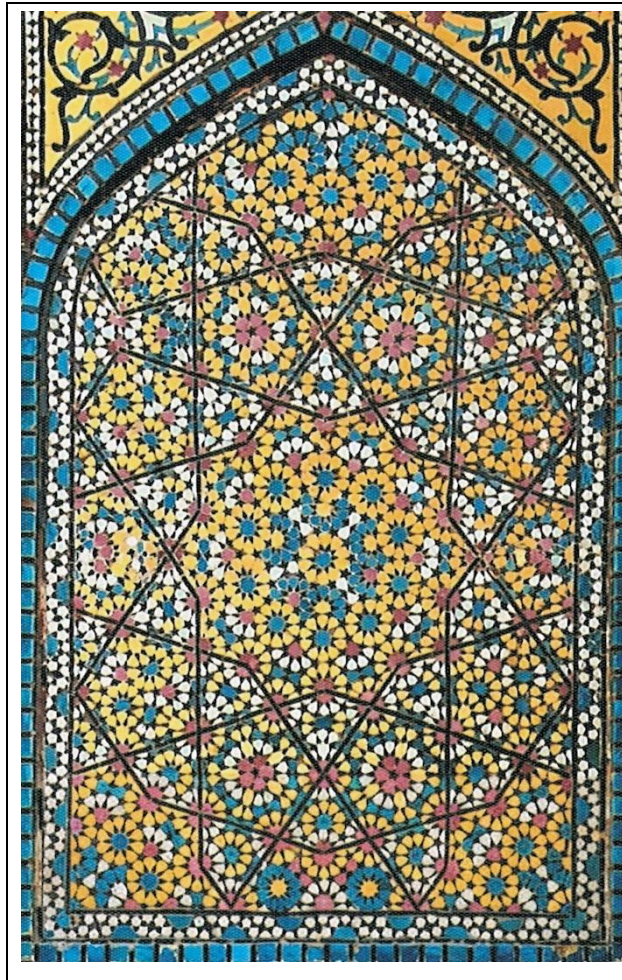


Fig. 2. Left – a wall mosaic from Hazrat-e Abdolazim Shrine in Shahr-e Rey, Iran

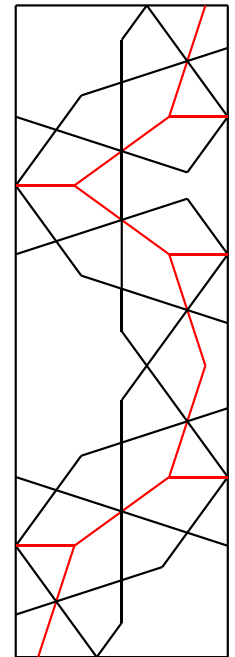
Photo by Fatemah Nazarian

Fig. 2a. Right – the main pattern and its tessellation

The photo's resolution is very low. So, we will have to guess parts of its geometry. A major issue is the central part of the big star. There are some design errors.

Another issue worth mentioning is the bottom edge of the stars placed in the corners and the top part of the design.

This mosaic is very unique. The main pattern (right) is a typical Nodir Devon Madrasah style design. Sometimes people use the Farsi term “*dah-e-kond*”. At the same time, the secondary pattern was created in the Kukeldash Madrasah style. It is often called “*dah-e-tond*” in Farsi. Thus, we will refer to this pattern as *kond & tond*.



After checking many geometric patterns with two layers, we can observe that they are mostly of type *kond & kond*, or *tond & kond*. Thus, reconstructing this pattern gives us the real chance to use the alternative tiles described in [8] and to expand our knowledge of them. There are a few more types of decagonal patterns worth considering, but this topic is beyond the scope of this paper.

GSP and GeoGebra comment

Patterns like the one discussed here contain thousands of geometric elements – points, segments, and polygons. Thus, we need to simplify our constructions and avoid all unnecessary elements. In all geometric patterns, points are essential. They are the fundamentals of the whole construction. Segments are secondary objects, and if there is no construction point on them, they can be deleted or hidden. Therefore, in all our constructions, we will use very few segments. Polygons are the objects used to display pattern elements. We need them. But one can still optimize their construction using non-framed polygons. This will significantly reduce file size and computer memory usage.

Matching secondary pattern elements around the edges of the main pattern

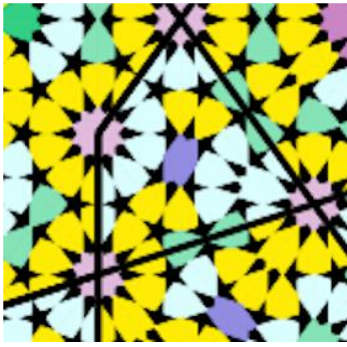


Fig. 3. The relation between main and secondary patterns

The drawing shows one of the main features of the two-level patterns. Edges of the main pattern serve as mirrors. Shapes on one side of the primary pattern lines are mirror images of those on the other side. Colors may differ, but shapes should remain the same. Only in rare situations, this match may be ignored. Therefore, the very first construction step will be to create a tessellation along these edges. The rest can be constructed as the user wants.

We will come back to this issue later.

Analysis of the main pattern

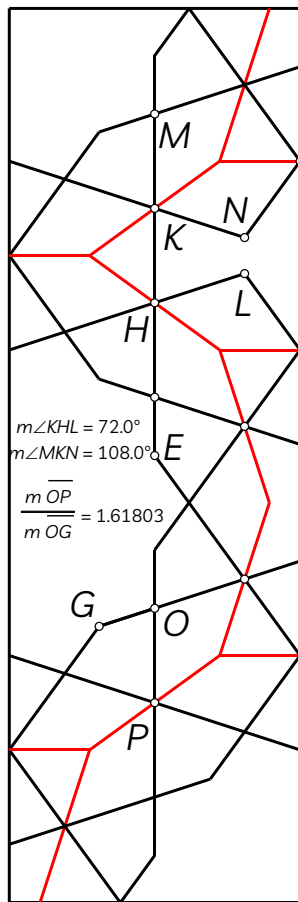


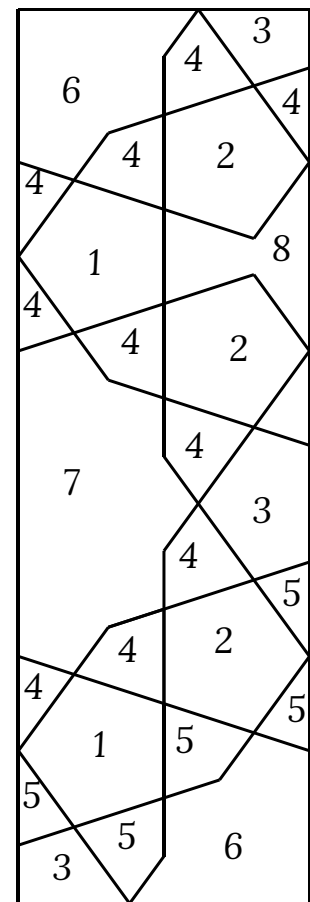
Fig. 4. Left – angles and segments of the main pattern

In the main pattern, we have only two angles: 72 and 108 degrees, and only two lengths of edges. The ratio of these two edges is $L/s = 1.61803$, and it is the well-known golden ratio we often see in decagonal patterns.

Fig. 4a. Right – map of polygons in the main pattern

In this whole design, we have regular pentagons 1 and 2. Their sizes are identical, but they differ in their secondary patterns. Kites 4 and 5 have different secondary patterns. Stars 6 and 7 are identical in size, but again, they have different secondary patterns. The central star has a very specific structure (and some errors), so it cannot be used as a pattern for corners.

Our construction will consist of creating a pattern for each of the polygons from 1 to 8 and pasting them into the right place.



GSP/GeoGebra tools for tessellations

In this case study, we will use the simplest possible tools for creating tessellations. Each tool starts with two points, shown in the following table in red. We use points exclusively. No polygons. As we deal here with alternative tessellations, blue segments will represent tessellation tiles. Each of these segments can be deleted later, after the pattern elements are placed. This way, we can reduce the number of components of the whole design.

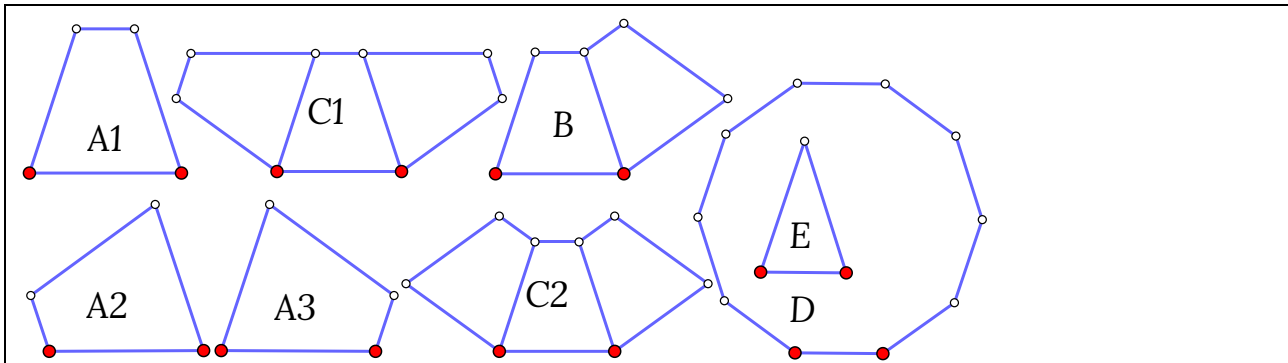


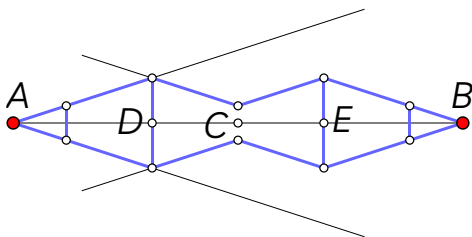
Fig. 5. Basic custom tools for constructing alternative tessellations

Each of these tools was created using rotations of one point about another. We used rotations about angles 72, -72, 144, or -144 degrees. No fancy constructions, rotations only. Figures B, C1, and C2 were created using tools A1, A2, and A3. Decagon D was also obtained by rotating points about an angle of 144 degrees. Triangle E is a 1/10 part of the decagon D.

NOTE: Each of these polygons was created by rotating or reflecting points. Blue segments were added to improve the readability of the tessellation. Thus, these segments can be deleted or hidden without destroying the final construction.

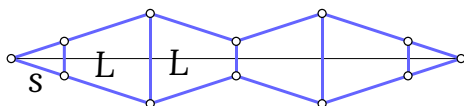
Constructing tessellations near the edges of the primary pattern

Tessellations along the edges of the primary pattern are the key elements of the whole design. They can be very simple or very elaborate. The pattern from Shahr-e Rey (see Fig. 2. uses a simple set of polygons, but this can be even simpler. As we said at the beginning, in the primary pattern, there are two segment lengths. Let us call them L and s . Here is the construction for the long segment.



Point C is the midpoint of AB. Points D and E are golden sections of AC and CB.

The tool uses the midpoint tool, the golden section tool, and rotations of the line by 18 degrees on each side of AB. For the final creation of the “edge tessellation” tool, we use only segments and points, shown here. Point C can also be hidden. Thus, the final tool will look like below.



Edge tessellation

In terms of the golden ratio, we can label this tessellation as $s4Ls$, where s and L are the lengths of polygons obtained using the golden section.

Labels and red color for starting points were used here for your information. In a real design, we should avoid using labels and colors for points. This issue is important. Otherwise, we will be struggling with a mess of colored points and labels that sometimes randomly appear. Before producing the final result, we may delete all segments of the tessellation.

After using this tool on a regular pentagon (one of the polygons in the primary pattern), we get the image:

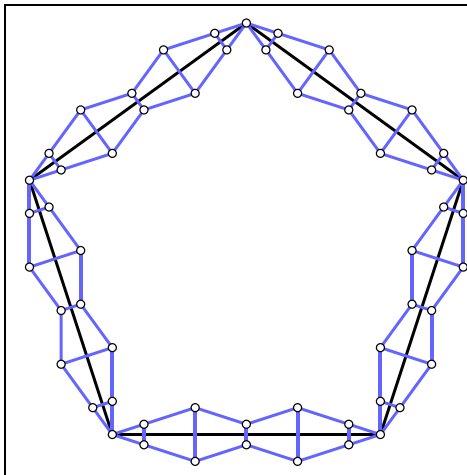


Fig. 6. A regular pentagon with its edge tessellation

The idea presented here shows us all the necessary polygons to start the secondary pattern. The space inside the pentagon can be filled with any convenient tessellation. As one might expect, we can invent a few decagonal alternative tessellations for this empty area. In the pattern from Shahr-e Rey (see Fig. 2. , we have two such fills.

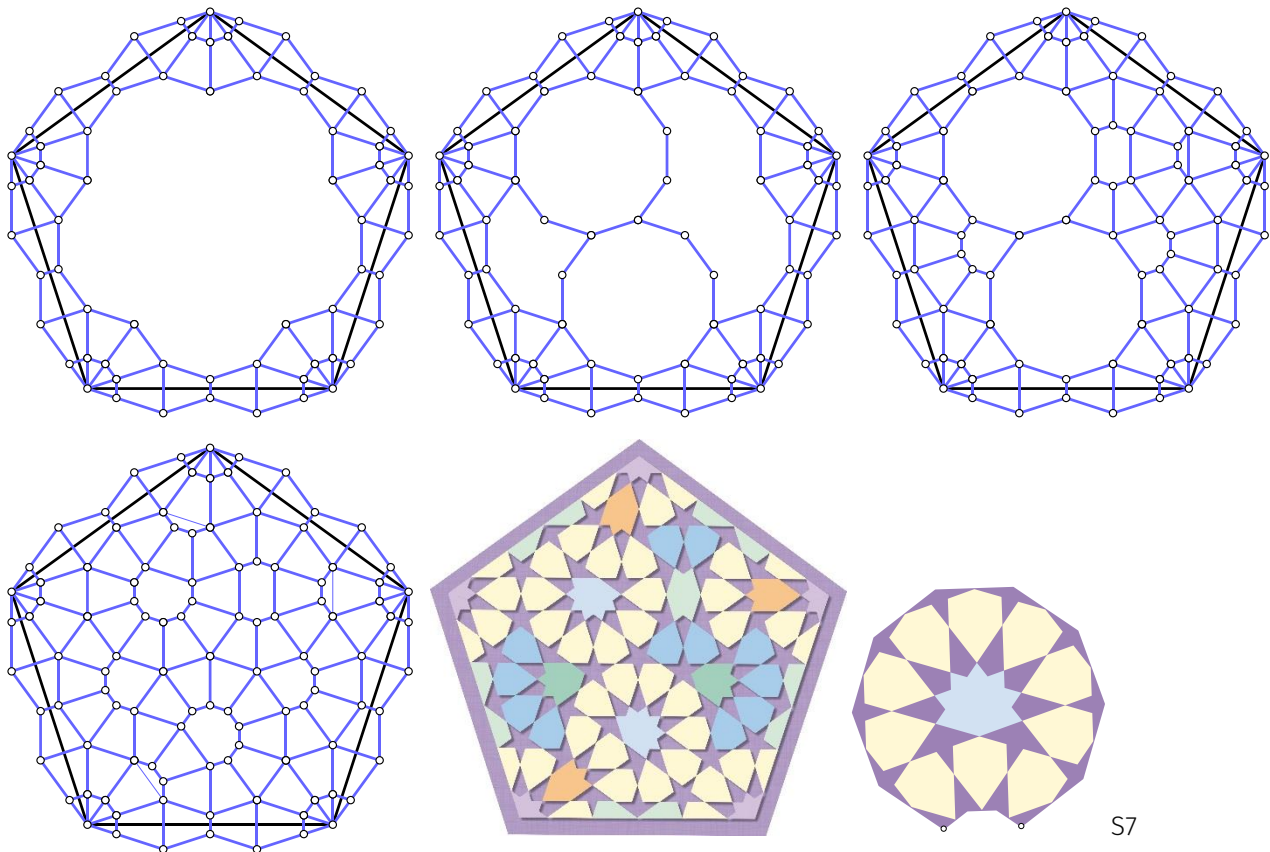
In the next pages, we will show edge tessellations for each of the primary pattern shapes.

NOTE: The white space inside this pentagon suggests using regular overlapping decagons for a future pattern.

Constructions of secondary patterns for each shape of the primary pattern

Now we can reconstruct secondary patterns and their tessellations for all shapes shown in the analysis of the primary pattern, i.e., polygons 1, ..., 8. We will start from pentagons.

Fig. 7. Pentagon 1

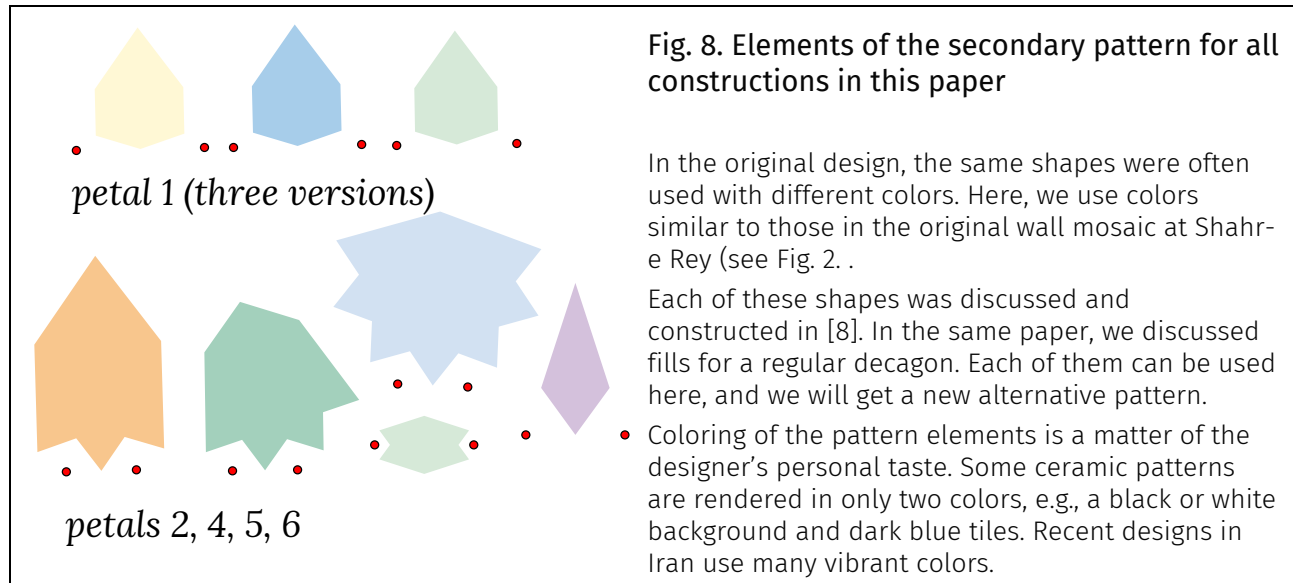


Steps for creating the secondary pattern for pentagon 1

Top row – a regular pentagon with its edge pattern. As we see, we can fit inside of it five overlapping decagons; fitting pattern for two of these decagons, this is the same as S7 described in [8]; fitting extra elements in the top-right decagon, this is the S8 shape from [8].

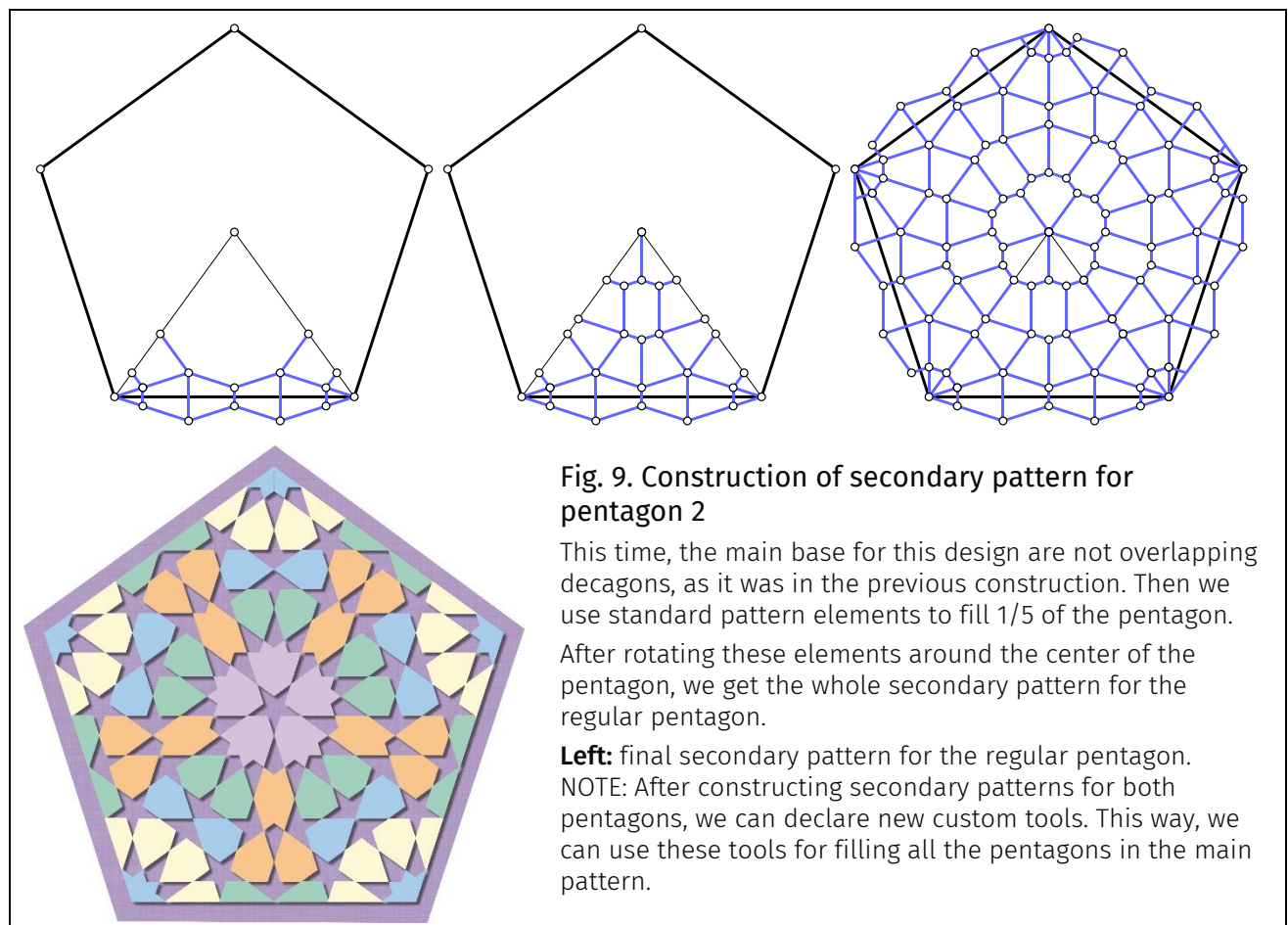
Bottom row – the final tessellation and the secondary pattern for the regular pentagon, and the S7 shape.

In this design, we used the pattern shapes discussed in [8], or this and the next constructions, we used the following pattern shapes.



Pattern for pentagon 2

The secondary pattern for pentagon 1 was asymmetric. But for pentagon 2, the designer created a secondary pattern with five symmetry lines. Therefore, its construction is much simpler.



Construction of the pattern for the corner star

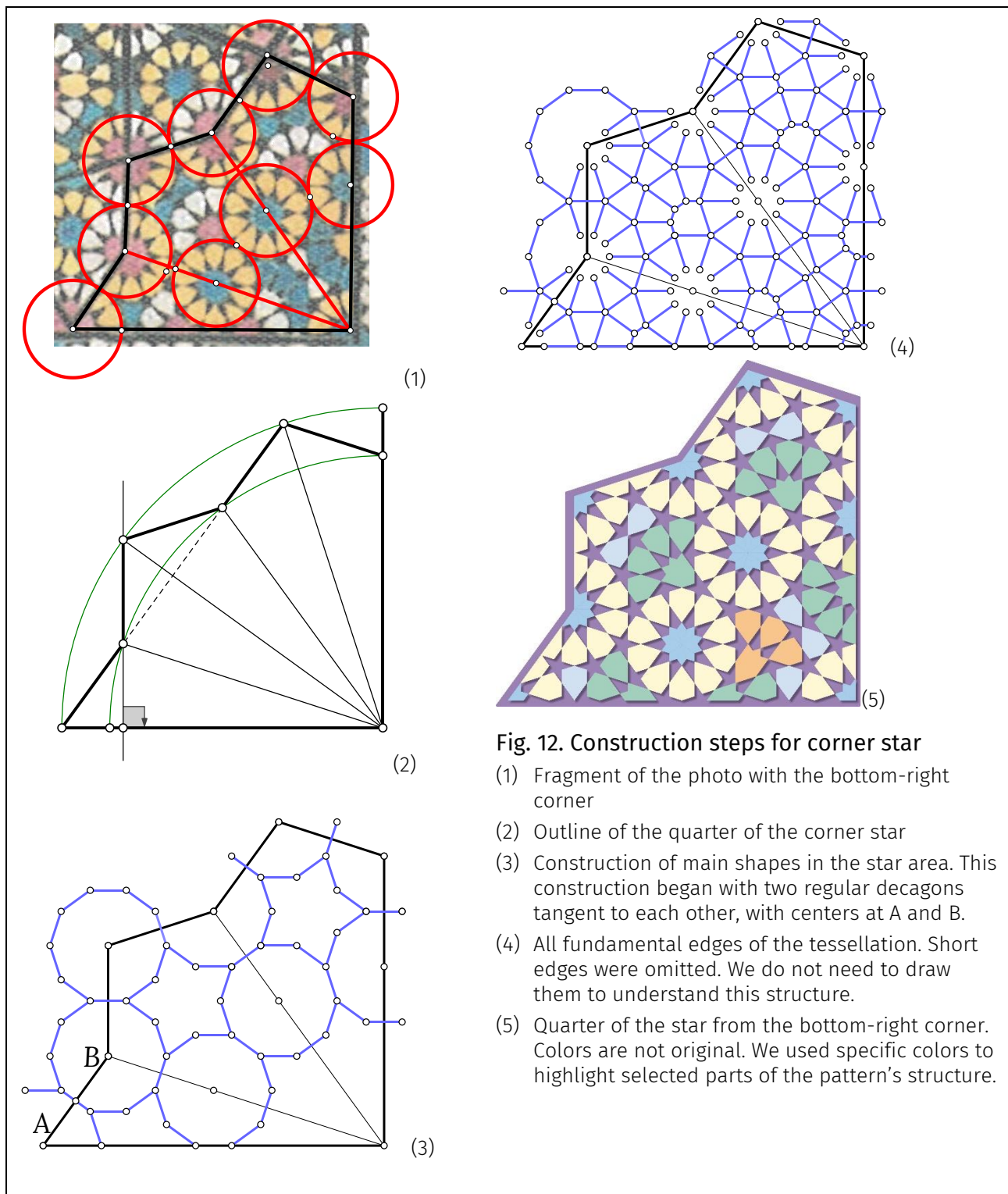


Fig. 12. Construction steps for corner star

- (1) Fragment of the photo with the bottom-right corner
- (2) Outline of the quarter of the corner star
- (3) Construction of main shapes in the star area. This construction began with two regular decagons tangent to each other, with centers at A and B.
- (4) All fundamental edges of the tessellation. Short edges were omitted. We do not need to draw them to understand this structure.
- (5) Quarter of the star from the bottom-right corner. Colors are not original. We used specific colors to highlight selected parts of the pattern's structure.

Secondary pattern for the central star

In the pattern from Shahr-e Rey (see Fig. 2.), the secondary pattern for the central star is different from that for the corner star. There are also some design errors, so we will have to construct a similar fill without repeating errors.

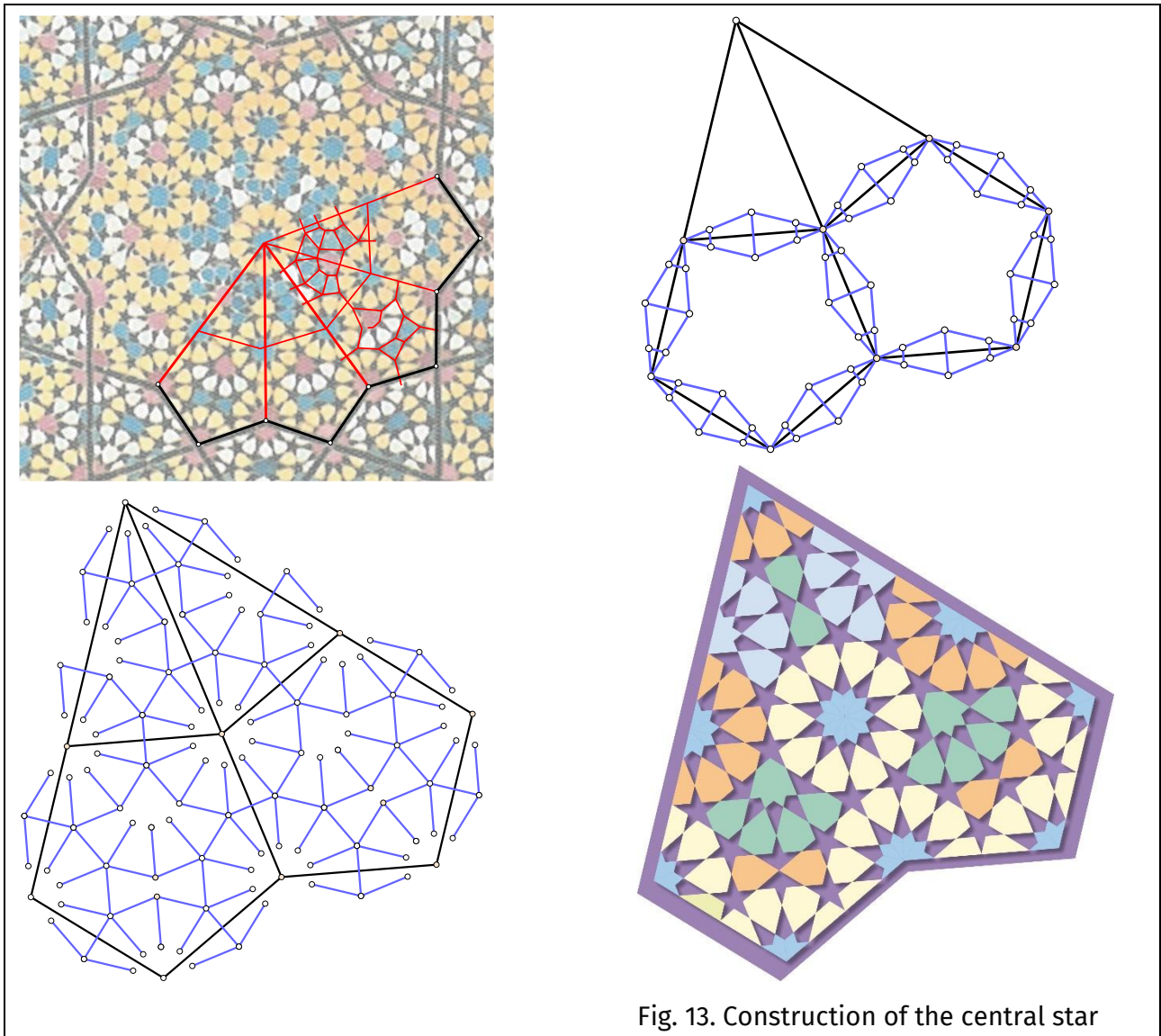


Fig. 13. Construction of the central star

Note that in the construction above, two long kites are combined into a single polygon. Neither of these two kites has a symmetry line. But the resulting shape has a symmetry line in the middle. Moreover, in all the tessellations created here, we avoided adding short edges. This way, we could make more readable tessellations.

Construction of the pattern for the flat pentagon

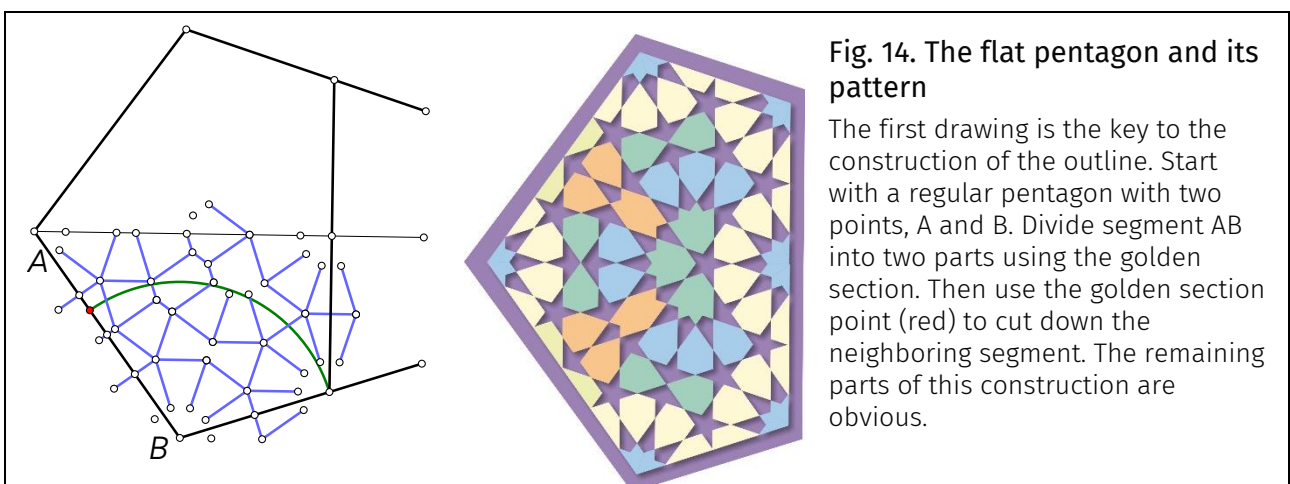
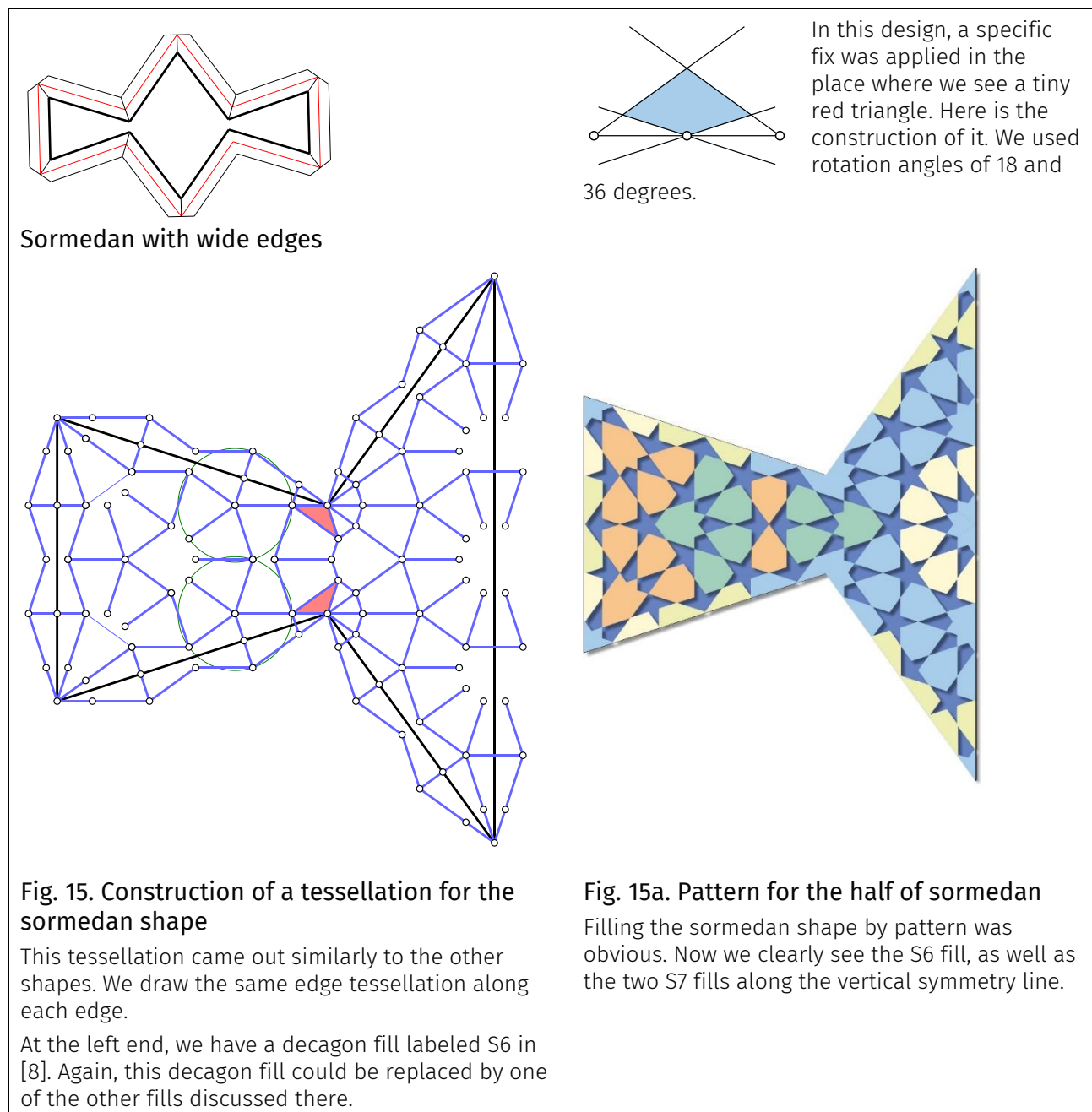


Fig. 14. The flat pentagon and its pattern

The first drawing is the key to the construction of the outline. Start with a regular pentagon with two points, A and B. Divide segment AB into two parts using the golden section. Then use the golden section point (red) to cut down the neighboring segment. The remaining parts of this construction are obvious.

Secondary pattern for the sormedan shape

In the main pattern, there is a specific shape, often called, from Persian, sormedan. Until now, most of the polygons in the main pattern were convex². Sormedan polygon is not convex. This feature causes trouble when making the primary pattern's lines thicker. Thicker lines change the size of the space in the narrow place inside the sormedan. The following drawing shows what will happen if we use sticks to create a real sormedan shape, here in red. The polygon inside such a construction will be narrower. Another bad feature of the sormedan with thicker edges is that its edge lengths will also change. In the geometric sormedan shown here, using red lines, all edges have the same length. In the black one inside, their lengths are different.



² Although flat polygon is convex as a single shape in most designs it is half of a non convex octagon. Thus it causes the same problems like the sormedan shape.

Sormedan was the last element for the final pattern. One can easily create a few other variations of the elements we created here. Here is a good opportunity to make something of our own.

Let us start assembling the whole design. Note: Each pattern we created in the previous pages should be saved as a custom tool. This way, we can easily fill all polygons of the primary pattern.

Construction of the primary pattern

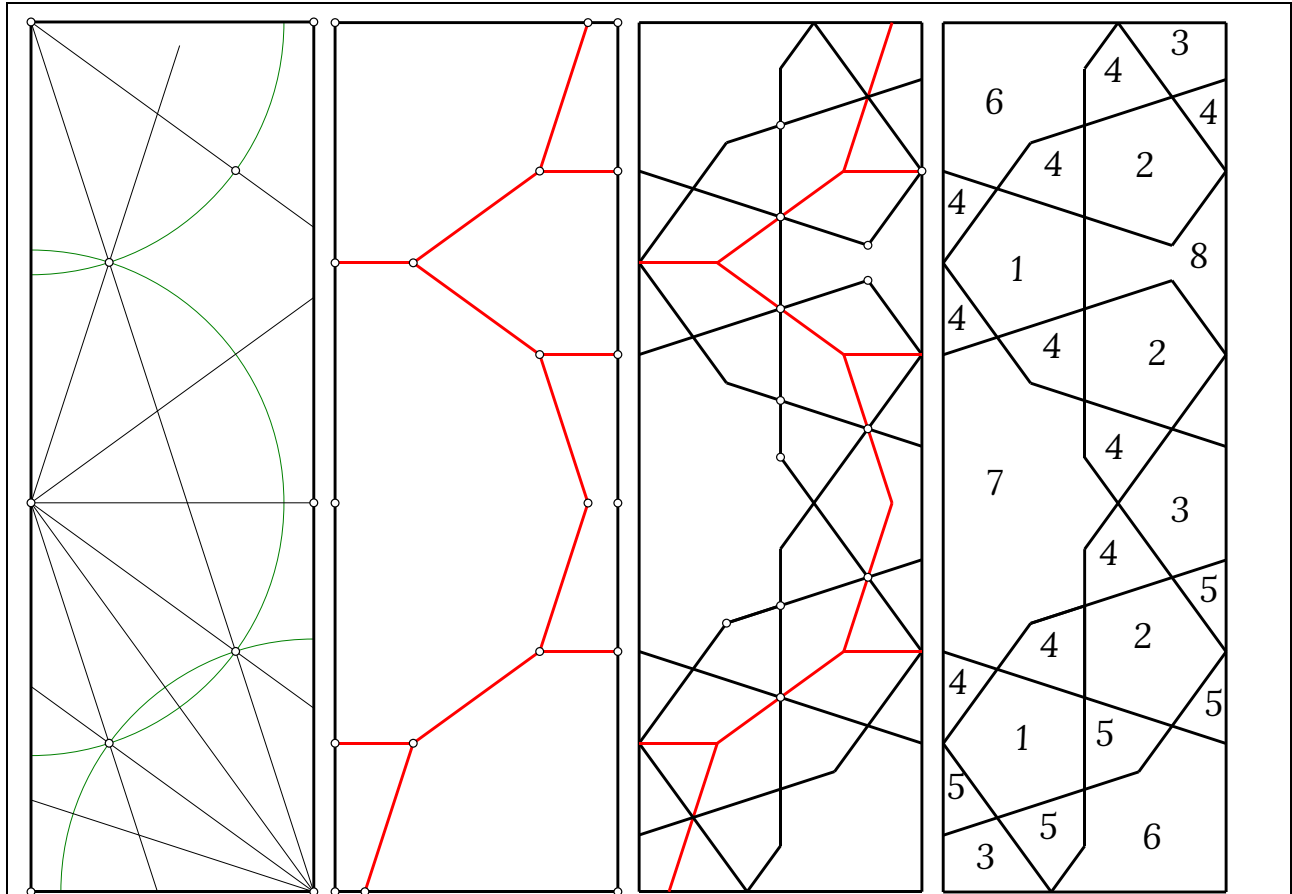


Fig. 16. Creation of the right side of the main pattern

The contour for the main pattern is a combination of two rectangles, as shown in the left drawing. Tessellation was formed by the intersections of arcs and lines that divide corners into five equal parts. The next drawing shows the tessellation and pattern created on it.

The right drawing presents the pattern map, with numbers indicating where each component should be placed.

The final reconstruction of the pattern from Shahr-e Rey (see Fig. 2.) is enclosed on the next page. The complete design was created in GSP, copied into Xara Designer, and enhanced by adding 3D edges to the primary pattern. Colors are not original, but Persian ceramic patterns can have many variations in color palettes. Note also – in this design, there are large pentagons with asymmetric patterns. In the original design, these patterns were oriented in a random direction. Thus, this part may differ slightly from the original design.

The ideas presented so far can be used to develop many other complex patterns. We need another tessellation, create another Persian-style pattern, and fill it with elements developed in this paper. If we meet new shapes along the way, we know how to create a Kukeldash Madrasah-style design for them. However, it may not always be a simple task.

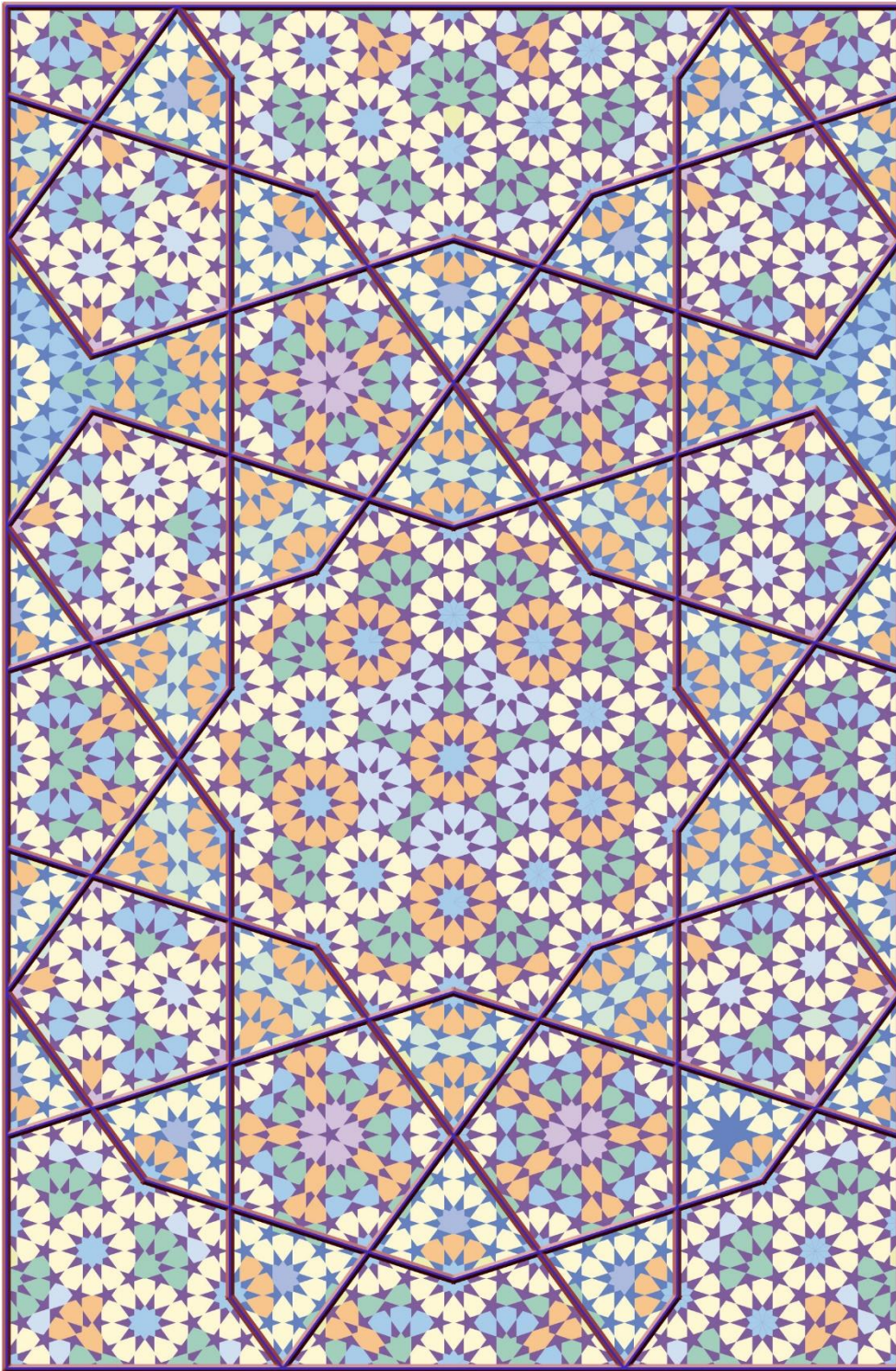


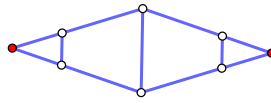
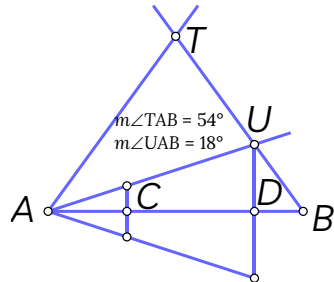
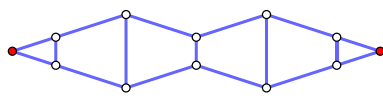
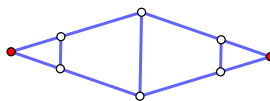
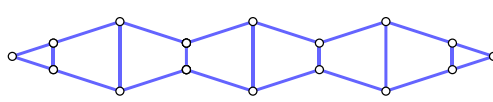
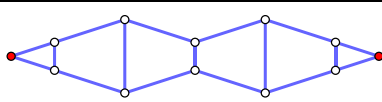
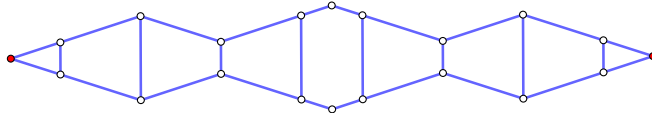
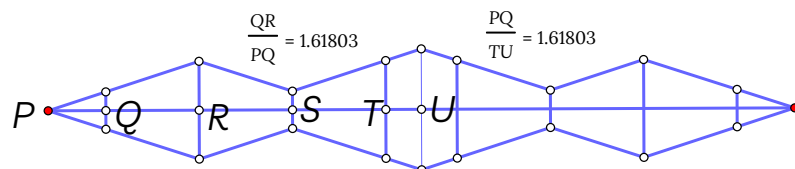
Fig. 17. Final reconstruction of the pattern from Shahr-e Rey

Where to go next

The idea presented here can be extended to other designs used as primary patterns.

We can also consider using either simpler or more elaborate edge tessellations. The edge of the pattern from Shahr-e Rey can be simplified. This way, we may get a less dense final design but a more difficult one to construct.

Fig. 18. More edge tessellations for the pattern from Shahr-e Rey (see Fig. 2.

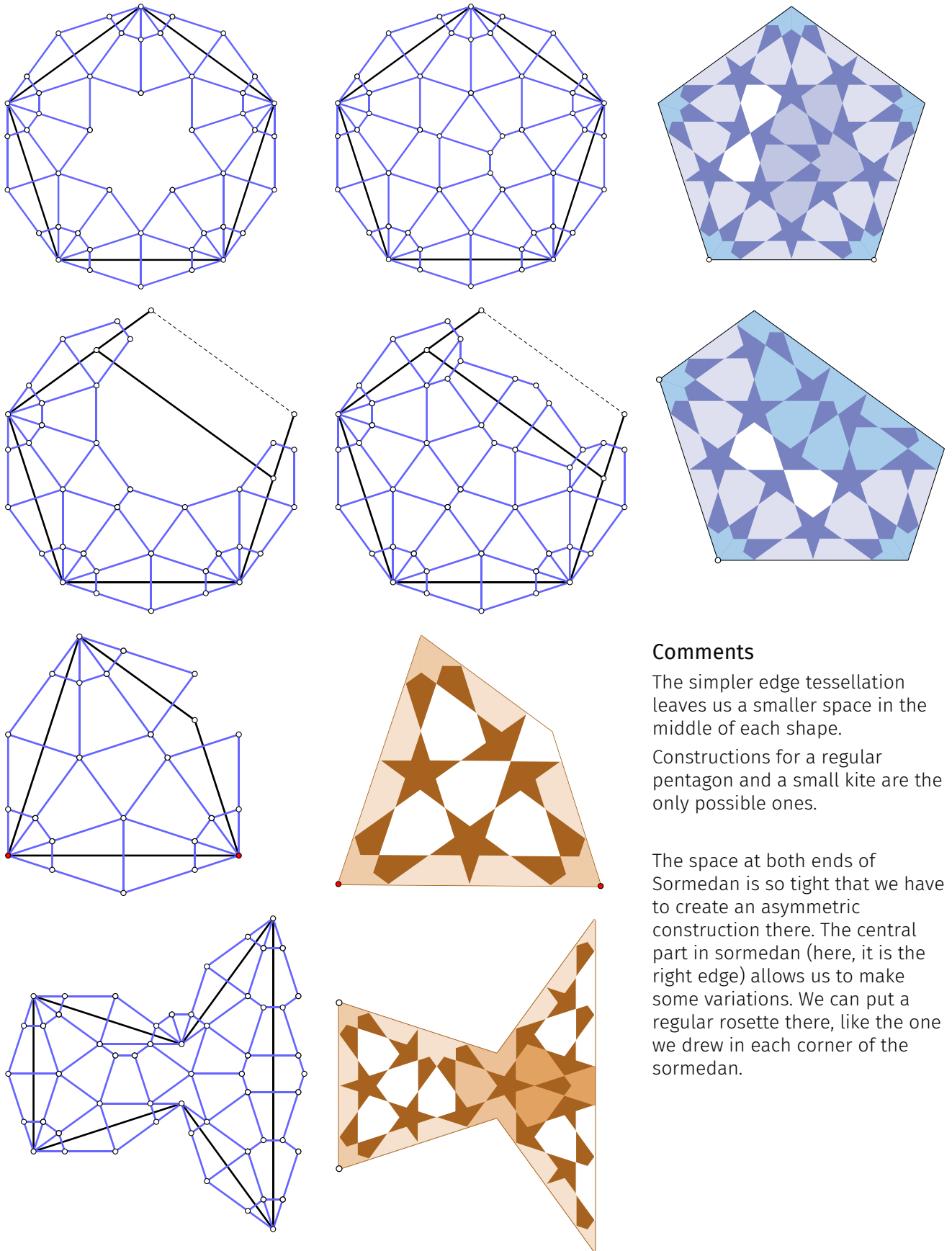
Long edge tessellation	Short edge tessellation	Description
 s2Ls		Edge tessellations for the pattern with lower complexity than the pattern from Shahr-e Rey. Point C is the golden section of AD. A and B are the edge endpoints.
 s4Ls		Edge tessellations for the original pattern from Shahr-e Rey
 s6Ls		Edge tessellations for a denser pattern than the Shahr-e Rey pattern.
 S3L2t3Ls		One can also create a more complex edge tessellation. What will the tessellation look like for the short edge?
NOTATION – In this table, we used notation like <i>s6Ls</i>, etc. In these drawings, we have three lengths: <i>t</i>, <i>s</i>, and <i>L</i>. These three numbers form a golden ratio sequence with <i>t</i> the smallest value, <i>s</i>, the medium value, and <i>L</i> the largest, and $L/s = s/t = 1.61803 \dots$. Thus, the formula <i>s6Ls</i> means the sequence of numbers <i>sLLLLLs</i>.		
		

Although we could create even more complex edge tessellations than those in the table, there is always the major question: can we produce proper fills for the polygons from the main pattern? Let us look at a simpler version of Shahr-e Rey's design (see Fig. 2. . Construction of this design is easier than the original Shahr-e Rey pattern. We deal here with less complex edge tessellations, and the areas inside the shapes of the primary pattern are much smaller. Thus, in some cases, we do not have much choice.

A reduced Shahr-e Rey pattern (compare Fig. 2.

We assume that the main pattern is the same as in the previous design. We use the edges tessellations as described in the first row of the above table: s2Ls (long edge tessellation), and sLt (short edge tessellation), i.e. $AC = s$, $CD = L$ and $DB = t$.

Fig. 19. Fills for the polygons of the main pattern



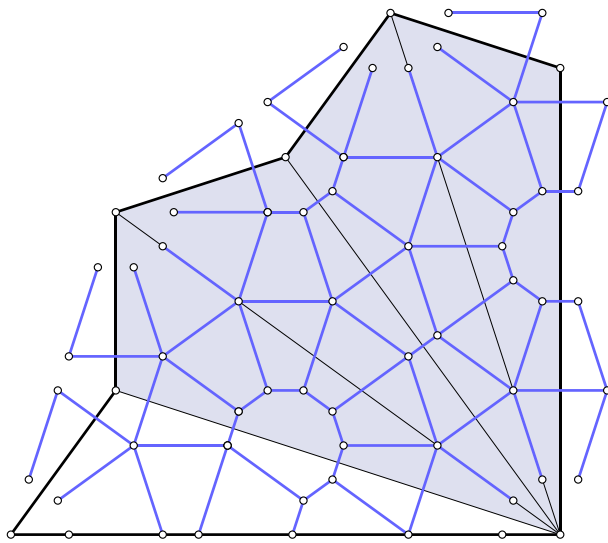
Comments

The simpler edge tessellation leaves us a smaller space in the middle of each shape.

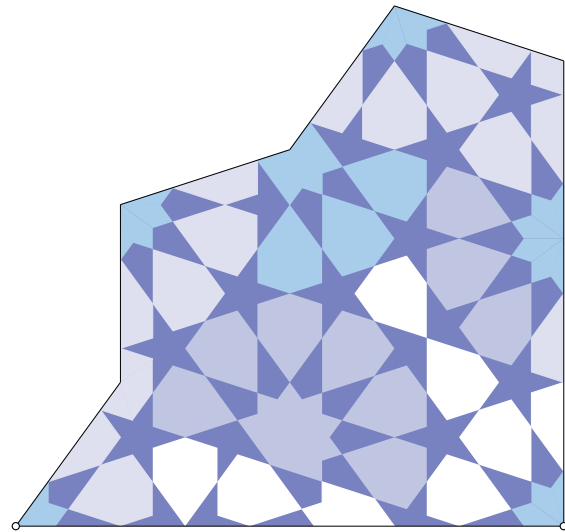
Constructions for a regular pentagon and a small kite are the only possible ones.

The space at both ends of Sormedan is so tight that we have to create an asymmetric construction there. The central part in sormedan (here, it is the right edge) allows us to make some variations. We can put a regular rosette there, like the one we drew in each corner of the sormedan.

Fig. 19a (continuation of the Fig. 19.)



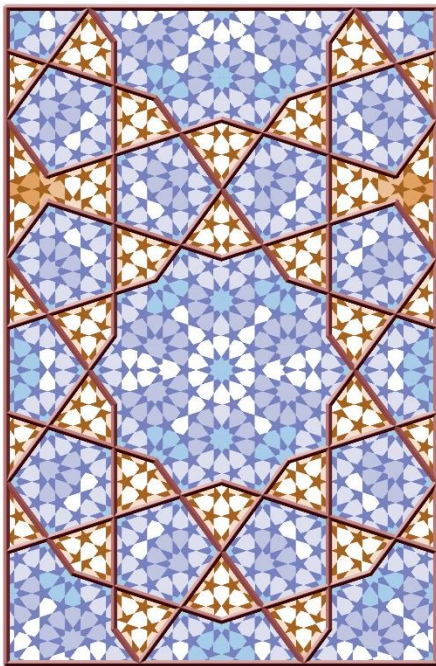
The central star and corner stars can be created using constructions like this one. The key for this particular construction is the shaded shape.



The space inside these stars is large, and we can make some variations. If we ignore the necessity of creating symmetric constructions, then we have even more freedom.

Fig. 20. A simplified version of the pattern from Shahr-e Rey (see Fig. 2).

The pattern is definitely less complex than the original Shahr-e Rey mosaic. But it is still very complex, and shapes for polygons of the first pattern can be reused in many other designs.



Two levels or embedded

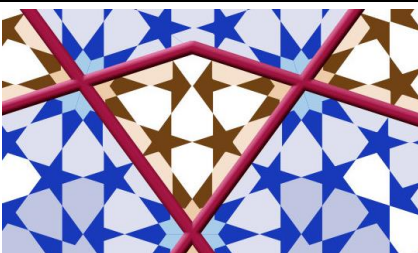
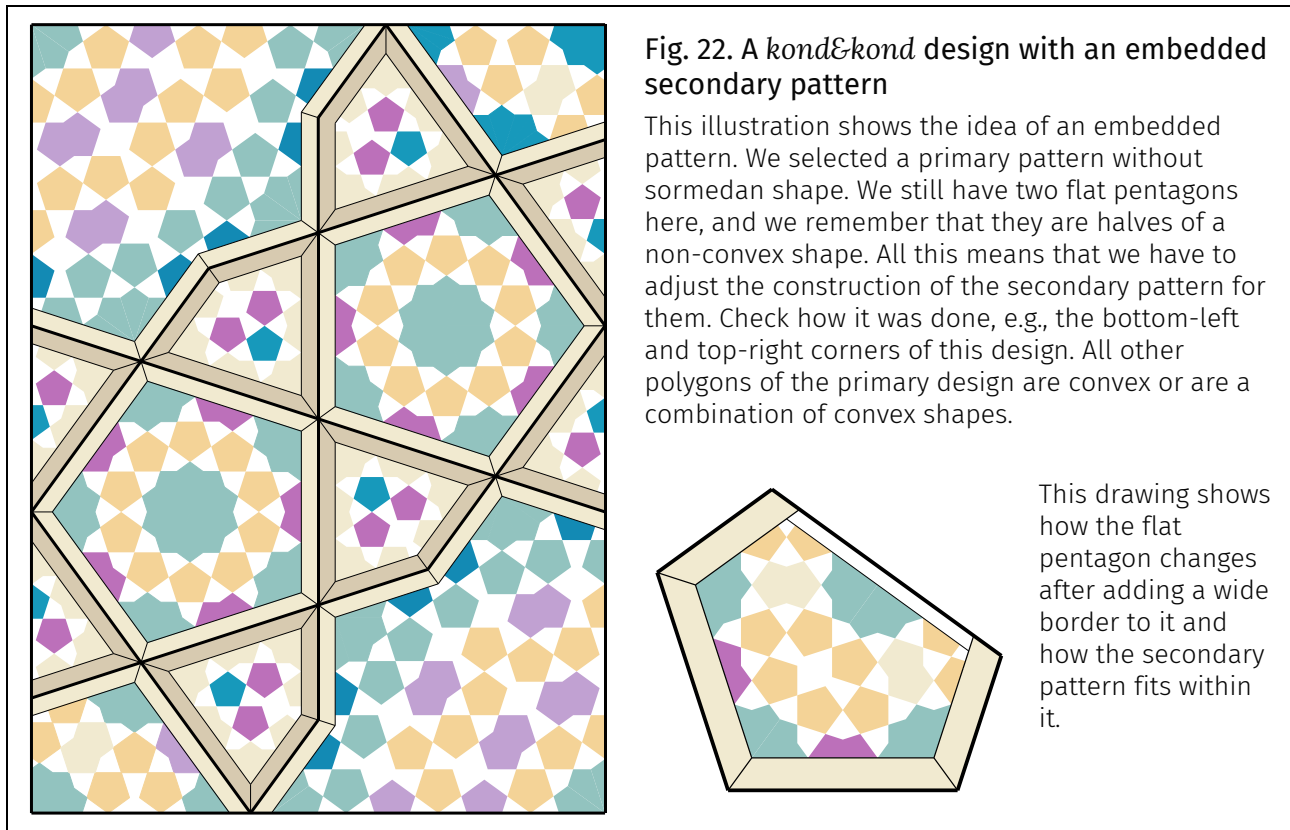


Fig. 21. A close look at the last design

The image shows a fragment of the last design. We can notice that the edges of the primary design overlap with the shapes of the secondary pattern. This fact means that if we want create these edges wider, some polygons in the secondary design may disappear completely. A construction like this may be called a two-level pattern, or, as some people call it, “a dual level pattern”.

Avoiding the problem mentioned above is possible, but we have to limit our designs to the primary patterns that contain only convex polygons.



Summary

In this paper, we examined a unique geometric pattern from Shahr-e Rey (see Fig. 2. . We examined how two patterns, one called primary and the other secondary, can fit together within the same geometric constructions.

We introduced the concept of edge tessellations and examined how two different edge tessellations can be used to construct very complex designs. In Persian geometric patterns, many similar constructions occur. This unique concept exists exclusively in Iran, with rare exceptions outside of it.

The drawing presented above is an example of a kond&kond design. Its geometric construction was created in the same way as all other geometric designs in this paper. We also used the concept of alternative tessellations.

Appendix: Tond, kond, and shol

Ideas presented in this text are related to a single example where we deal with the main pattern made in Nodir Devon Madrasah style (in Farsi *dah-e-kond*, or according to Tony Lee type 1), and the secondary pattern was made in Kukeldash Madrasah style (in Farsi *dah-e-tond*, or according to Tony Lee classification type II). But the typology of patterns and their combinations can be more complex than we can discuss in detail here. In brief, we can consider primary patterns and secondary patterns to be the same type or two different types.

In general, we can identify three main groups of decagonal patterns in Iranian architecture, along with a few minor groups and patterns that mix features of the mentioned groups. Let us discuss them briefly.

Kukeldash Madrasah style

This group is often described in Farsi as *dah-e-tond*. It is a group with a distinctive petal shape. In my papers, I use the term Kukeldash Madrasah style. For a Western reader, this name is easier to remember. In Tony Lee's works, this group is referred to as type 2.

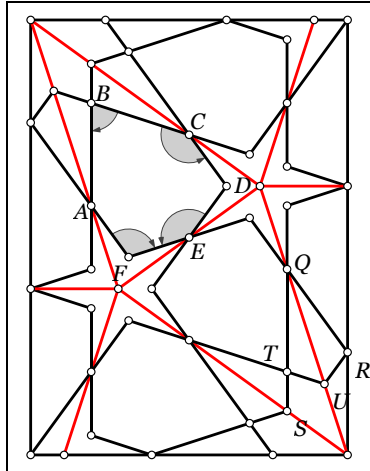


Fig. 23. dah-e-tond or Kukeldash Madrasah style, or type 2 (by Tony Lee)

$$\begin{aligned} m\angle CBA &= 72.00^\circ & m\angle AFE &= 108.00^\circ \\ m\angle FED &= 144.00^\circ & m\angle TQR &= 36.00^\circ \\ m\angle UTS &= 72.00^\circ & \frac{m\overline{BC}}{m\overline{CD}} &= 1.61803 \end{aligned}$$

A specific feature of this style are angles 144 and 36 degrees. Its petal has a nice shape, with two edges parallel, and a ratio of the long edge to the short edge equal to the golden ratio. In this type of pattern, we have stars with sharp edges and 36-degree angles.

Nodir Devon Madrasah style

This is the type of pattern that was used as the primary pattern in our study.

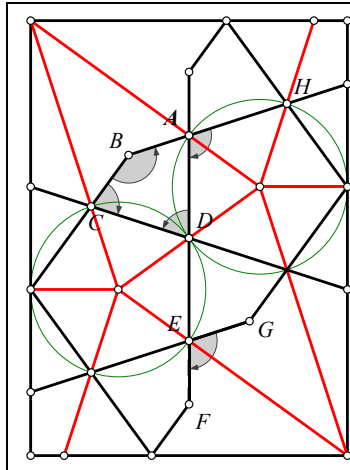


Fig. 24. Dah-e-kond or Nodir Devon Madrasah style, or type 1 (by Tony Lee)

A specific feature in this pattern are regular pentagons.

Angles:

$$\begin{aligned} m\angle ABC &= 144.00^\circ & m\angle HAD &= 108.00^\circ \\ m\angle BCD &= 72.00^\circ & m\angle GEF &= 108.00^\circ \\ \frac{DC}{CB} &= 1.61803 \end{aligned}$$

Here we also meet the golden ratio.

Persian style

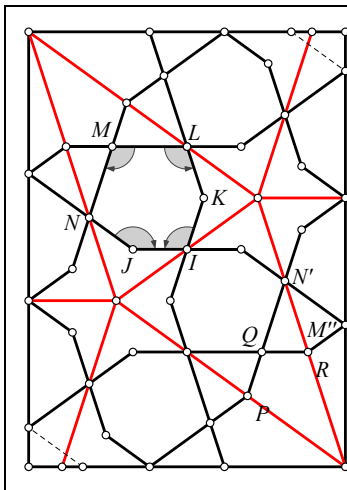


Fig. 25. Persian style, or dah-e-shol in Farsi, or type III according to Tony Lee

This type of pattern is also known as the median ten. It is the type intermediate between the two other types described above. A pentagonal star evolves from a star with a sharp edge to a regular pentagon. It uses angles:

$$\begin{aligned} m\angle JIK &= 108.00^\circ & m\angle IJN &= 144.00^\circ & \frac{m \overline{MN}}{m \overline{NJ}} &= 1.38197 \\ m\angle LMN &= 108.00^\circ & m\angle M''N'Q &= 72.00^\circ & & \\ m\angle MLK &= 108.00^\circ & & & & \end{aligned}$$

The petal in this pattern is different from the one in the Kukeldash Madrasah style; long edges are parallel to short edges, and there is no golden ratio of its edges.

This brief information about the three types of decagonal patterns suggests that we may create at least nine kinds of two-level or embedded patterns. By considering different edge tessellations, the number of such patterns can grow significantly. Constructions of some of these patterns were described in [4].

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All drawings were made by the Author using Geometer's Sketchpad (GSP). GSP is a free software, and it can be downloaded through the web page: <https://symmetrica.wordpress.com/gsp-2/>

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